

## ON EXTENDABILITY OF PERMUTATIONS

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ABSTRACT. Let  $V$  be a left vector space over a division ring and let  $\mathcal{P}(V)$  be the associated projective space. We describe all finite subsets  $X \subset V$  such that every permutation on  $X$  can be extended to a linear automorphism of  $V$  and all finite subsets  $\mathcal{X} \subset \mathcal{P}(V)$  such that every permutation on  $\mathcal{X}$  can be extended to an element of  $\mathrm{PGL}(V)$ . Also, we reformulate the results in terms of linear and projective representations of symmetric groups.

## 1. INTRODUCTION

Let  $V$  be a left vector space over a division ring  $R$ . We suppose that  $\dim V = n$  is finite and not less than 2. Denote by  $\mathcal{P}(V)$  the associated projective space formed by 1-dimensional subspaces of  $V$ .

Our first result (Theorem 1) is related to extendability of permutations on finite subsets of  $V$  to linear automorphisms of  $V$ : if every permutation on a finite subset of  $V$  can be extended to a linear automorphism of  $V$  then this subset is formed by linearly independent vectors or it consists of

$$x_1, \dots, x_m, -(x_1 + \dots + x_m),$$

where  $x_1, \dots, x_m$  are linearly independent vectors.

Under the assumption that  $R$  is a field, we determine all finite subsets  $\mathcal{X} \subset \mathcal{P}(V)$  such that every permutation on  $\mathcal{X}$  can be extended to an element of  $\mathrm{PGL}(V)$ . Our second result (Theorem 2) states that there are precisely three distinct types of such subsets.

The main results (Theorems 1 and 2) will be reformulated in terms of linear and projective representations of symmetric groups (Corollaries 1 and 2).

## 2. PERMUTATIONS ON FINITE SUBSETS OF VECTOR SPACES

Let  $X$  be a finite subset of  $V$  containing more than one vector. Denote by  $S(X)$  the group of all permutations on  $X$ . We want to determine all cases when every element of  $S(X)$  can be extended to a linear automorphism of  $V$ . This is possible, for example, if  $X$  is formed by linearly independent vectors.

**Example 1.** Suppose that  $X$  consists of linearly independent vectors  $x_1, \dots, x_m$  and the vector

$$x_{m+1} = -(x_1 + \dots + x_m).$$

For every  $i \in \{1, \dots, m-1\}$  we take any linear automorphism  $u_i \in \mathrm{GL}(V)$  such that

$$u_i(x_i) = x_{i+1}, \quad u_i(x_{i+1}) = x_i \quad \text{and} \quad u_i(x_j) = x_j \quad \text{if } j \neq i, i+1, m+1.$$

Every  $u_i$  sends  $x_{m+1}$  to itself. Consider a linear automorphism  $v \in \text{GL}(V)$  leaving fixed every  $x_i$  for  $i \leq m-1$  and transferring  $x_m$  to  $x_{m+1}$ . Then

$$v(x_{m+1}) = -(v(x_1) + \cdots + v(x_m)) = -(x_1 + \cdots + x_{m-1} - x_1 - \cdots - x_{m-1} - x_m) = x_m.$$

So, all transpositions of type  $(x_i, x_{i+1})$  can be extended to linear automorphisms of  $V$ . Since  $S(X)$  is spanned by these transpositions, every permutation on  $X$  is extendable to a linear automorphism of  $V$ .

**Theorem 1.** *If every permutation on  $X$  can be extended to a linear automorphism of  $V$  then  $X$  is formed by linearly independent vectors or it consists of*

$$x_1, \dots, x_m, -(x_1 + \cdots + x_m),$$

where  $x_1, \dots, x_m$  are linearly independent.

*Proof.* Let  $x_1, \dots, x_k$  be the elements of  $X$ . Suppose that these vectors are not linearly independent and consider any maximal collection of linearly independent vectors from  $X$ . We can assume that this collection is formed by  $x_1, \dots, x_m$ ,  $m < k$ . Then every  $x_p$  with  $p > m$  is a linear combination of  $x_1, \dots, x_m$ , i.e.  $x_p = \sum_{l=1}^m a_l x_l$ . Let  $u \in \text{GL}(V)$  be an extension of the transposition  $(x_i, x_j)$ ,  $i, j \leq m$ . Then

$$u(x_i) = x_j, u(x_j) = x_i \text{ and } u(x_l) = x_l \text{ if } l \neq i, j.$$

We have

$$\sum_{l=1}^m a_l x_l = x_p = u(x_p) = \sum_{l=1}^m b_l x_l, \text{ where } b_i = a_j, b_j = a_i \text{ and } b_l = a_l \text{ if } l \neq i, j.$$

Since  $x_1, \dots, x_m$  are linearly independent, the latter means that  $a_i = a_j$ . This equality holds for any  $i, j \leq m$  and we have

$$x_p = a(x_1 + \cdots + x_m)$$

for some non-zero scalar  $a \in R$ . Let  $v \in \text{GL}(V)$  be an extension of the transposition  $(x_1, x_p)$ . Then

$$v(x_1) = x_p, v(x_p) = x_1 \text{ and } v(x_i) = x_i \text{ if } i \neq 1, p$$

We have

$$\begin{aligned} x_1 &= v(x_p) = a(v(x_1) + \cdots + v(x_m)) = a(x_p + x_2 + \cdots + x_m) = \\ &= a^2(x_1 + \cdots + x_m) + a(x_2 + \cdots + x_m) = a^2 x_1 + (a^2 + a)(x_2 + \cdots + x_m). \end{aligned}$$

Hence  $a^2 = 1$  and  $a^2 + a = 0$  which implies that  $a = -1$  and

$$x_p = -(x_1 + \cdots + x_m).$$

This equality holds for every  $p > m$ . Therefore,  $k = m+1$  and the second possibility is realized.  $\square$

Let  $X$  be a finite subset of  $V$  such that every permutation on  $X$  can be extended to a linear automorphism of  $V$ . Suppose that  $|X| \geq 2$  and  $\langle X \rangle = V$ . Then for every  $s \in S(X)$  there is the unique extension  $\alpha_X(s) \in \text{GL}(V)$ . If  $s, t \in S(X)$  then  $\alpha_X(st)$  and  $\alpha_X(s)\alpha_X(t)$  both are extensions of  $st$  which guarantees that

$$\alpha_X(st) = \alpha_X(s)\alpha_X(t).$$

Thus  $\alpha_X$  is a monomorphism of  $S(X)$  to  $\text{GL}(V)$  (it is clear that the kernel of  $\alpha_X$  is trivial). The image of  $\alpha_X$  will be denoted  $G(X)$ .

**Corollary 1.** *Let  $G$  be a subgroup of  $\text{GL}(V)$  isomorphic to  $S_m$ . Let also  $X$  be an orbit of  $G$  such that  $G$  acts faithfully on  $X$  and  $|X| = m$ <sup>1</sup>. Suppose that there are not proper  $G$ -invariant subspaces of  $V$ . Then the following assertions are fulfilled:*

- $X$  is a base of  $V$  or  $X = \{x_1, \dots, x_n, -(x_1 + \dots + x_n)\}$ , where  $x_1, \dots, x_n$  form a base of  $V$ ;
- $G = G(X)$ .

*Proof.* Let  $r$  be the homomorphism of  $G$  to  $S(X)$  transferring every  $g \in G$  to  $g|_X$ . Since the action of  $G$  on  $X$  is faithful,  $r$  is a monomorphism. It follows from our assumptions that  $G$  and  $S(X)$  have the same number of elements. Thus  $r$  is an isomorphism which means that every permutation on  $X$  can be extended to a linear automorphism of  $V$ . Since  $\langle X \rangle$  is  $G$ -invariant, we have  $\langle X \rangle = V$ . Then  $r^{-1} = \alpha_X$  and  $G = G(X)$ .  $\square$

### 3. PERMUTATIONS ON FINITE SUBSETS OF PROJECTIVE SPACES

Let  $\mathcal{X}$  be a finite subset of  $\mathcal{P}(V)$  containing more than one element. Denote by  $S(\mathcal{X})$  the group of all permutations on  $\mathcal{X}$ . In this section we determine all cases when every element of  $S(\mathcal{X})$  can be extended to an element of  $\text{PGL}(V)$  (if  $R$  is a field).

Recall that the group  $\text{PGL}(V)$  is formed by the transformations of  $\mathcal{P}(V)$  induced by linear automorphisms of  $V$ . Let  $\pi$  be the natural homomorphism of  $\text{GL}(V)$  to  $\text{PGL}(V)$ . The kernel of  $\pi$  consists of all homotheties  $x \rightarrow ax$ , where  $a$  belongs to the center of  $R$ , i.e. two linear automorphisms of  $V$  induce the same element of  $\text{PGL}(V)$  if and only if one of them is a scalar multiple of the other.

We say that  $P_1, \dots, P_m \in \mathcal{P}(V)$  form an *independent* subset if non-zero vectors  $x_1 \in P_1, \dots, x_m \in P_m$  are linearly independent. Every permutation on an independent subset can be extended to an element of  $\text{PGL}(V)$ .

Let  $m \in \{2, \dots, n\}$ . An  $(m+1)$ -element subset  $\mathcal{X} \subset \mathcal{P}(V)$  is called an  *$m$ -simplex* if it is not independent and every  $m$ -element subset of  $\mathcal{X}$  is independent. For example, if  $x_1, \dots, x_m \in V$  are linearly independent and  $a_1, \dots, a_m \in R$  are non-zero then

$$\langle x_1 \rangle, \dots, \langle x_m \rangle \text{ and } \langle a_1 x_1 + \dots + a_m x_m \rangle$$

form an  $m$ -simplex. Conversely, if  $\{P_1, \dots, P_{m+1}\}$  is an  $m$ -simplex then there exist linearly independent vectors

$$x_1 \in P_1 \setminus \{0\}, \dots, x_m \in P_m \setminus \{0\} \text{ such that } P_{m+1} = \langle x_1 + \dots + x_m \rangle.$$

Every permutation on an  $m$ -simplex can be extended to an element of  $\text{PGL}(V)$  [1, Section III.3, Proposition 1].

Following [1, Section III.4, Remark 5] we say that a subset  $\mathcal{X} \subset \mathcal{P}(V)$  is *harmonic* if there are linearly independent vectors  $x, y \in V$  such that

$$\mathcal{X} = \{ \langle x \rangle, \langle y \rangle, \langle x + y \rangle, \langle x - y \rangle \}.$$

**Example 2.** Suppose that the characteristic of  $R$  is equal to 3 and  $\mathcal{X}$  is the harmonic subset consisting of

$$P_1 = \langle x \rangle, P_2 = \langle y \rangle, P_3 = \langle x + y \rangle, P_4 = \langle x - y \rangle.$$

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<sup>1</sup>If  $X$  is an orbit of  $G$  and  $G$  acts faithfully on  $X$  then  $|X| \geq m$ .

Consider  $u_1, u_2, u_3 \in \text{GL}(V)$  satisfying the following conditions

$$\begin{aligned} u_1(x) &= y & u_1(y) &= x, \\ u_2(x) &= -x & u_2(y) &= x + y, \\ u_3(x) &= x & u_3(y) &= -y. \end{aligned}$$

Since the characteristic of  $R$  is equal to 3, we have

$$u_2(x - y) = -x - (x + y) = -2x - y = x - y.$$

A direct verification shows that every  $\pi(u_i)$  is an extension of the transposition  $(P_i, P_{i+1})$ . Since the group  $S(\mathcal{X})$  is spanned by all transpositions of type  $(P_i, P_{i+1})$ , every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(V)$ .

**Theorem 2.** *Suppose that  $R$  is a field. If every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(V)$  then one of the following possibilities is realized:*

- $\mathcal{X}$  is an independent subset;
- $\mathcal{X}$  is an  $m$ -simplex,  $m \in \{2, \dots, n\}$ ;
- the characteristic of  $R$  is equal to 3 and  $\mathcal{X}$  is a harmonic subset.

**Lemma 1.** *Suppose that  $R$  is a field. Let  $f$  be an element of  $\text{PGL}(V)$  transferring  $P \in \mathcal{P}(V)$  to  $Q \in \mathcal{P}(V)$ . For any non-zero vectors  $x \in P$  and  $y \in Q$  there exists  $u \in \text{GL}(V)$  such that  $\pi(u) = f$  and  $u(x) = y$ .*

*Proof.* We take any  $v \in \text{GL}(V)$  such that  $\pi(v) = f$ . Then  $v(x) = ay$  and the linear automorphism  $u := a^{-1}v$  is as required.  $\square$

**Remark 1.** If  $R$  is non-commutative then a scalar multiple of a linear mapping is linear only in the case when the scalar belongs to the center of  $R$ .

*Proof of Theorem 2.* Let  $P_1, \dots, P_k$  be the elements of  $\mathcal{X}$ . If  $\mathcal{X}$  is not independent then we take any maximal independent subset in  $\mathcal{X}$ . Suppose that it is formed by  $P_1, \dots, P_m$ ,  $k < m$  and consider  $P_p$  with  $p > m$ . Every non-zero vector  $y \in P_p$  is a linear combination of non-zero vectors  $y_1 \in P_1, \dots, y_m \in P_m$ . If this linear combination contains  $y_i$  and does not contain  $y_j$  for some  $i, j \leq m$  then an element of  $\text{PGL}(V)$  extending the transposition  $(P_i, P_j)$  does not leave fixed  $P_p$  which is impossible. This means that

$$y = a_1 y_1 + \dots + a_m y_m,$$

where all  $a_1, \dots, a_m \in R$  are non-zero.

Thus  $P_1, \dots, P_m$  and  $P_p$  form an  $m$ -simplex for every  $p > m$ . If  $\mathcal{X}$  consists of  $m+1$  elements, i.e.  $k = m+1$ , then  $\mathcal{X}$  is an  $m$ -simplex. Consider the case when  $k \geq m+2$ .

We choose non-zero vectors  $x_1 \in P_1, \dots, x_m \in P_m$  such that

$$x_{m+1} := x_1 + \dots + x_m \in P_{m+1}.$$

If  $p \geq m+2$  then

$$P_p = \langle x_p \rangle, \text{ where } x_p = b_1 x_1 + \dots + b_m x_m$$

and all  $b_1, \dots, b_m \in R$  are non-zero. Let  $v$  be a linear automorphism of  $V$  such that  $\pi(v)$  is an extension of the transposition  $(P_{m+1}, P_p)$ . By Lemma 1, we can suppose that  $v$  sends  $x_{m+1}$  to  $x_p$ . Since  $x_1, \dots, x_m$  are linearly independent and  $v(P_i) = P_i$  for every  $i \leq m$ , the equality

$$v(x_1) + \dots + v(x_m) = v(x_{m+1}) = x_p = b_1 x_1 + \dots + b_m x_m$$

shows that  $v(x_i) = b_i x_i$  for every  $i \leq m$ . Then

$$v(x_p) = b_1 v(x_1) + \cdots + b_m v(x_m) = b_1^2 x_1 + \cdots + b_m^2 x_m \in P_{m+1}$$

which means that  $b_1^2 = b_2^2 = \cdots = b_m^2$  and  $b_i = \pm b_j$  for any  $i, j \leq m$ . In other words,

$$x_p = b(\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m),$$

where  $\varepsilon_i = \pm 1$  for every  $i \in \{1, \dots, m\}$ . Since  $x_{m+1}$  and  $x_p$  are linearly independent,  $\varepsilon_i \neq \varepsilon_j$  for some pairs  $i, j \leq m$ . This guarantees that the characteristic of  $R$  is not equal to 2 and we can assume that

$$P_p = \langle x_p \rangle, \text{ where } x_p = x_1 + \cdots + x_q - x_{q+1} - \cdots - x_m$$

and  $1 \leq q < m$ .

Now consider a linear automorphism  $u \in \text{GL}(V)$  such that  $\pi(u)$  is an extension of the transposition  $(P_q, P_{q+1})$ . Then

$$u(P_q) = P_{q+1}, u(P_{q+1}) = P_q \text{ and } u(P_i) = P_i \text{ if } i \neq q, q+1.$$

By Lemma 1, we suppose that  $u$  leaves fixed  $x_{m+1}$ . Since  $x_1, \dots, x_m$  are linearly independent, the equality

$$u(x_1) + \cdots + u(x_m) = u(x_{m+1}) = x_{m+1} = x_1 + \cdots + x_m$$

implies that

$$u(x_q) = x_{q+1}, u(x_{q+1}) = x_q \text{ and } u(x_i) = x_i \text{ if } i \neq q, q+1, i \leq m.$$

Then

$$u(x_p) = u(x_1) + \cdots + u(x_q) - u(x_{q+1}) - \cdots - u(x_m)$$

belongs to  $P_p$  only in the case when  $q = 1$  and  $m = 2$ , i.e.  $P_p = \langle x_1 - x_2 \rangle$  for every  $p \geq 4$ . The latter means that  $\mathcal{X}$  is the harmonic subset consisting of

$$P_1 = \langle x_1 \rangle, P_2 = \langle x_2 \rangle, P_3 = \langle x_1 + x_2 \rangle, P_4 = \langle x_1 - x_2 \rangle.$$

Let  $w$  be a linear automorphism of  $V$  such that  $\pi(w)$  is an extension of the transposition  $(P_1, P_3)$  and  $w(x_1) = x_1 + x_2$  (see Lemma 1). Since  $w(P_2) = P_2$  and  $w(P_3) = P_1$ , we have

$$w(x_1 + x_2) = w(x_1) + w(x_2) = (x_1 + x_2) + cx_2 \in P_1.$$

Then  $c = -1$  and  $w(x_2) = -x_2$ . The equality  $w(P_4) = P_4$  implies that

$$w(x_1 - x_2) = w(x_1) - w(x_2) = (x_1 + x_2) + x_2 = x_1 + 2x_2 \in P_4.$$

Hence  $x_1 + 2x_2 = x_1 - x_2$  and  $2 = -1$ , i.e. the characteristic of  $R$  is equal to 3.  $\square$

Every representation  $\alpha : S_m \rightarrow \text{GL}(V)$  induces the projective representation  $\pi\alpha : S_m \rightarrow \text{PGL}(V)$ . By [3], there exist projective representations of symmetric groups which are not induced by linear representations (an explicit realization of such representations can be found in [2]). Now we establish an analogue of Corollary 1 for projective representations of  $S_m$ .

Let  $\mathcal{X}$  be a subset of  $\mathcal{P}(V)$  such that every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(V)$ . Suppose that  $|\mathcal{X}| \geq 2$  and there is not a proper subspace of  $V$  containing every element of  $\mathcal{X}$ . The following example shows that an extension of a permutation on  $\mathcal{X}$  to an element of  $\text{PGL}(V)$  is not unique if  $\mathcal{X}$  is a maximal independent subset (an independent subset consisting of  $n$  elements).

**Example 3.** Let  $x_1, \dots, x_n$  be a base of  $V$  and let  $a_1, \dots, a_n$  be distinct non-zero scalars. Consider the linear automorphism of  $V$  transferring every  $x_i$  to  $a_i x_i$ . The associated element of  $\mathrm{PGL}(V)$  is non-trivial, but it induces the identity permutation on the set consisting of  $\langle x_1 \rangle, \dots, \langle x_n \rangle$ .

**Proposition 1.** Let  $\{P_1, \dots, P_{n+1}\}$  and  $\{P'_1, \dots, P'_{n+1}\}$  be  $n$ -simplices in  $\mathcal{P}(V)$ . The following two conditions are equivalent:

- $R$  is a field,
- there is a unique element of  $\mathrm{PGL}(V)$  transferring every  $P_i$  to  $P'_i$ .

*Proof.* See [1, Section III.3]. □

If  $R$  is a field and  $\mathcal{X} = \{P_1, \dots, P_{n+1}\}$  is an  $n$ -simplex then, by Proposition 1, for every permutation  $s \in S(\mathcal{X})$  there is the unique extension  $\bar{s} \in \mathrm{PGL}(V)$ . This correspondence is a monomorphism of  $S(\mathcal{X})$  to  $\mathrm{PGL}(V)$ . Its image will be denoted by  $G(\mathcal{X})$ . Note that  $G(\mathcal{X}) = \pi(G(X))$ , where  $X$  is formed by vectors

$$x_1 \in P_1, \dots, x_n \in P_n \text{ and } -(x_1 + \dots + x_n) \in P_{n+1}.$$

Suppose that  $R$  is a field of characteristic 3 and  $\mathcal{X}$  is a harmonic subset. Then every 3-element subset of  $\mathcal{X}$  is a 2-simplex and Proposition 1 guarantees that every permutation on  $\mathcal{X}$  is uniquely extendable to an element of  $\mathrm{PGL}(V)$ . As above, we get a monomorphism of  $S(\mathcal{X})$  to  $\mathrm{PGL}(V)$  and denote its image by  $G(\mathcal{X})$ .

**Corollary 2.** Let  $R$  be a field and let  $G$  be a subgroup of  $\mathrm{PGL}(V)$  isomorphic to  $S_m$ . Let also  $\mathcal{X}$  be an orbit of  $G$  such that  $G$  acts faithfully on  $\mathcal{X}$  and  $|\mathcal{X}| = m^2$ . Suppose that there are not proper  $G$ -invariant subspaces of  $V$ <sup>3</sup>. Then the following assertions are fulfilled:

- $\mathcal{X}$  is a maximal independent subset or an  $n$ -simplex or  $\mathcal{X}$  is a harmonic subset and the characteristic of  $R$  is equal to 3;
- if  $\mathcal{X}$  is not independent then  $G = G(\mathcal{X})$ .

The proof is similar to the proof of Corollary 1.

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<sup>2</sup>As in Corollary 1, if  $\mathcal{X}$  is an orbit of  $G$  and  $G$  acts faithfully on  $\mathcal{X}$  then  $|\mathcal{X}| \geq m$ .

<sup>3</sup>A subspace  $S \subset V$  is  $G$ -invariant if every element of  $G$  transfers  $\mathcal{P}(S)$  to itself.